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Surface oscillations of electromagnetically levitated viscous metal droplets

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We investigate the oscillation spectrum of electromagnetically levitated metal droplets. In the case of electromagnetic levitation, gravity is compensated by a Lorentz force, which is generated by an external current. The oscillation spectrum contains information about the thermophysical properties of the liquid metal, namely surface tension and viscosity. For a correct interpretation of these spectra the influence of the external forces on the frequencies and the damping of the surface waves must be well understood. The external forces deform the droplet, so that the static equilibrium shape is aspherical. For a perfect conductor the effect of the Lorentz force and gravity on the oscillation spectrum is calculated for an arbitrary magnetic field and arbitrary values of the viscosity. The high Reynolds number limit is evaluated. Explicit results are obtained for a linear magnetic field, which describes the experimental situation well.

1. Introduction

Surface oscillations of liquid droplets have been investigated with respect to very different subjects (Collins, Plesset & Saffren 1974). In 1879 Lord Rayleigh calculated the oscillation spectrum of a force-free and inviscid liquid droplet around its spherical equilibrium shape. The surface eigenmodes have the geometry of pure spherical harmonics and are $(2l+1)$-fold degenerate. The frequencies are

$$\nu_R(l) = \frac{1}{2\pi}\left(\frac{\gamma l(l+2)(l-1)}{\rho a^3}\right)^{1/2}, \tag{1}$$

with the density ρ, radius a and surface tension γ. The Rayleigh frequencies $\nu_R(l)$ are proportional to the square root of the surface tension. This suggests the idea of measuring the surface tension via the frequencies of an oscillating droplet. Because the viscosity μ was neglected, the surface modes are not damped in the calculation of Lord Rayleigh.

In 1961 Chandrasekhar calculated the viscous damping of a force free oscillating droplet. Now the complex frequencies σ of the eigenmodes are the roots of transcendental equations:

$$\left(\frac{l\gamma}{a^3\rho}(l+2)(l-1)+\sigma^2\right) - \left(\frac{2\,(l^2-1)}{a^2/c^2-2a/cQ_1(l,a/c)} + \frac{2l(l-1)}{a^2/c^2}\left[1-\frac{(l+1)Q_1(l,a/c)}{a/(2c)-Q_1(l,a/c)}\right]\right)\sigma^2 = 0 \tag{2}$$

with

$$Q_1(l, a/c) = \frac{J_{l+3/2}(a/c)}{J_{l+1/2}(a/c)}, \quad c = \left[\mu/(\sigma\rho)\right]^{1/2}, \quad \sigma = \mathrm{i}\omega + \Gamma,$$

where the J_i are the Bessel functions of order i. Because of the spherical symmetry the spectrum is still degenerate. Chandrasekhar evaluated this condition for low and for high Reynolds numbers. The intermediate regime was investigated by Suryanarayana & Bayazitoglu (1991*b*). In the high Reynolds number limit, which follows from $Re = a^2/c^2 \rightarrow \infty$, the complex frequencies are

$$\sigma(l) = \pm \mathrm{i}2\pi\nu_R(l) + \Gamma_C, \qquad \Gamma_C = \frac{\mu}{\rho a^2}(l-1)(2l+1). \tag{3}$$

The damping of the modes is proportional to the viscosity of the liquid. This suggests the idea of measuring the viscosity from the damping of the surface modes.

In the case of electromagnetic levitation a Lorentz force is generated by an external current compensating gravity (Okress *et al.* 1952; Lohöfer 1989, 1993). For levitation an inhomogenous electromagnetic field is required. It is not justified to neglect these strong forces.

In the liquid state the shape of the droplet is aspherical due to gravity and Lorentz force. About this equilibrium shape surface oscillations can be observed. Warham (1988) used a Lagrange formalism to study the influence of the external forces on the oscillation spectrum of an inviscid, electromagnetically levitated metal droplet. Cummings & Blackburn (1991) treated the same subject directly from the equations of motion. A closely related work was published by Suryanarayana & Bayazitoglu (1991*a*). The metal was assumed to be a perfect conductor. The static equilibrium shape of the droplet was investigated analytically (Cummings & Blackburn 1991) and numerically (Schwartz *et al.* 1993). Because the external forces destroy the spherical symmetry the degeneracy in the oscillation spectrum is removed. The geometry of the surface eigenmodes has changed, i.e. they are no longer pure spherical harmonics. The frequencies of the oscillations are solutions of a coupled set of equations:

$$\left(\frac{\gamma}{a^3\rho} l(l+2)(l-1) + \sigma^2\right)\xi_{lm} + M_{lm,uv}\xi_{uv} = 0, \tag{4}$$

where ξ_{lm} are the coefficients of the different spherical harmonics and the summation convention is adopted. The summation convention means that summation is performed for every index appearing twice. The matrix M, which follows from equation 2.57 of Cummings & Blackburn (1991), describes the influence of the external forces. We will neglect the off-diagonal elements of M as Cummings & Blackburn did, because they give only higher-order corrections to the perturbation analysis which will be performed.

Besides this the oscillation spectrum may be affected by a rotation of the droplet. This effect, which was studied by Busse (1984), will not be taken into account here.

In this paper we present a calculation of the oscillation spectrum of electromagnetically levitated viscous metal droplets, and for the first time viscosity and external forces are both taken into account. The work is closely related to that of Chandrasekhar (1961) and Cummings and Blackburn (1991). In practice, the rotational part of the Lorentz force will drive a laminar and stationary or a turbulent motion of the fluid (Sneyd & Moffat 1982; Mestel 1982). This motion is not related to the surface oscillations. Because we want to investigate these surface oscillations and avoid the computational problems associated with the stationary motion we assume the metal to be a perfect conductor, as others did (Warham 1988; Cummings & Blackburn 1991; Suryanarayana & Bayazitoglu 1991*a*). Then the skin depth is zero and the magnetic field cannot penetrate the droplet. Without a driving force no stationary motion is generated and the surface oscillations occur around a static equilibrium. It should be mentioned, however, that the perfect conductor approximation is difficult to justify. For a liquid metal the skin depth δ is calculated from $\delta = [2/(\mu_0 \kappa \omega)]^{1/2}$ (Jackson 1962), where κ is the conductivity and ω is the frequency of the magnetic field. With $\kappa \approx 0.05\,(\mu\Omega)^{-1}\,\mathrm{cm}^{-1}$ for liquid Cu at the melting point (Iida & Guthrie 1988), $\nu = 440$ kHz (Sauerland, Lohöfer & Egry 1993) we obtain $\delta \approx 10^{-4}$m. For a droplet radius of $a = 3 \times 10^{-3}$m the ratio $\delta/a \approx 0.1$ is relatively large for an expansion parameter to be considered only up to the zeroth order. The perfect conductor approximation is supported by the numerical investigation of El-Kaddah and Szekely (1983).

The magnitude of the static deformation depends on the ratio of the Lorentz force and the surface tension. For levitation of a 1 g metal droplet a magnetic field of approximatly 10^{-2}T is required. For such a field strength the ratio of the magnetic pressure $B^2/2\mu_0$ acting on the surface (Shercliff 1965) and the surface tension pressure γ/a, with $\gamma \approx 1\,\mathrm{Nm}^{-1}$, is $B^2a/2\mu_0\gamma \approx 0.1$. The ratio between the pressure contribution due to gravity and to surface tension is of the same order, because gravity and Lorentz force compensate each other. Therefore we expect the deformation to be in the range of a few percent of the spherical radius. The ratio $B^2a/2\mu_0\gamma$ is in fact the expansion parameter of the perturbation analysis. This ratio enters the calculation through the required pressure equilibrium at the surface. To make the expansion parameter visible throughout the whole calculation we introduce a formal expansion parameter ϵ by setting $B^2/\mu_0 \to \epsilon B^2/\mu_0$ and $g \to \epsilon g$. Calculating the spectrum of the surface oscillations to first order in ϵ is the same as calculating it to first order in the external forces. The final result is obtained by setting $\epsilon = 1$. It follows that the external forces must be calculated for oscillations around the spherical shape only, because the static deformation will be proportional to ϵ, too. Pressure, stress tensor and surface tension must be evaluated for oscillations around the aspherical shape.

We investigate this for an arbitrary magnetic field and arbitrary values for the viscosity. For surface oscillations on liquid metal droplets the Reynolds number *Re* is high. As a typical velocity in the problem we choose the velocity associated with the $l = 2$ modes from Rayleigh's formula (1). From

$$Re = \frac{a^2\rho}{\mu}\left(\frac{8\gamma}{\rho a^3}\right)^{1/2}$$

we obtain with a density $\rho \approx 5\,\mathrm{g\,cm^{-3}}$ and a viscosity $\mu \approx 1\,\mathrm{m\,Pa\,s}$ a Reynolds number $Re \approx 10^4$. Therefore we investigate the high Reynolds number limit. As an explicit realization we study the case of a linear magnetic field, as did others before (Cummings & Blackburn 1991; Suryanarayana & Bayazitoglu 1991*a*), because it is a good approximation to the experimental situation (Sauerland 1993).

The paper is organized as follows. In §2 the equations and boundary conditions that constitute the problem are presented. In the following §3 we list the required results, that were derived by Cummings and Blackburn (1991). These are the magnetic pressure and the static deformation. The flow field is related to the pressure oscillations, which occur as a consequence of the surface oscillations, through the boundary conditions on the tangential components of the stress vector in §4. With this and the boundary condition on the normal component of the stresses acting on the surface, the dynamic pressure is related to the amplitude of the surface oscillation in §5. The results are put into the kinematic boundary condition in the next section (§6) to yield an eigenvalue equation for the complex frequencies of the surface modes. The high Reynolds number limit is evaluated in §7, where explicit results for the frequencies and the damping are derived for an arbitrary magnetic field. The above-mentioned case of a linear magnetic field is evaluated in §8. We conclude with a short discussion (§9).

2. Formulation of the problem

In the perfect conductor approximation the problem splits into two parts, which can be solved as follows. First the magnetic field and from this the magnetic pressure must be calculated. Because the magnetic field vanishes inside the droplet we write

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}_o(\boldsymbol{r}) H(F(\boldsymbol{r})) \tag{5}$$

with

$$F(\boldsymbol{r}) = r - a(1 + (\xi_{lm} + R_{lm})\, Y_l^m(\Omega)), \tag{6}$$

where $H(x)$ is the step-function and $F(\boldsymbol{r}) = 0$ is the surface of the droplet. Here and in the rest of paper we used spherical coordinates (r, Θ, φ) and $\mathrm{d}\Omega$ is the usual abbreviation for the differential $\sin(\Theta)\mathrm{d}\Theta\mathrm{d}\varphi$. The coefficients R_{lm} are the amplitudes of the static deformation and the coefficients ξ_{lm} are the amplitudes of the dynamic deformation. Θ is the angle from the z-axis, which is chosen antiparallel to the direction of the gravity vector. The current density $\boldsymbol{j}(\boldsymbol{r})$ vanishes outside the droplet. Hence it follows from

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \mu_0 \boldsymbol{j}(\boldsymbol{r}), \tag{7}$$

that the magnetic field must satisfy

$$\nabla \times \boldsymbol{B}_o(\boldsymbol{r}) = 0 \tag{8}$$

and that the current density is given as a surface current density

$$\mu_0 \boldsymbol{j}(\boldsymbol{r}) = \delta(F(\boldsymbol{r})) \nabla F(\boldsymbol{r}) \times \boldsymbol{B}_o(\boldsymbol{r}), \tag{9}$$

where $\delta(x)$ is the Dirac δ-function. From this it follows that the Lorentz force density

$$\boldsymbol{F}_L = \boldsymbol{j} \times \boldsymbol{B} \tag{10}$$

is zero everywhere except on the surface. From

$$\nabla \cdot \boldsymbol{B} = 0 \tag{11}$$

follows the continuity of the normal component of the magnetic field across the surface and hence, because of (5), the normal component vanishes on the surface

$$\boldsymbol{B}_o \cdot \boldsymbol{n}|_{F=0} = 0, \tag{12}$$

where

$$\boldsymbol{n} = \left.\frac{\nabla F}{|\nabla F|}\right|_{F=0} \tag{13}$$

is the unit normal vector directed outwards. The magnetic field is completely determined through equations (8), (11) and (12). From (9) and (12) it follows that the Lorentz force is always directed normal to the surface and hence can be written as a magnetic pressure (Cummings & Blackburn 1991)

$$p_{MAG} = \epsilon \left. \frac{B_0^2}{2\mu_0} \right|_{F=0} \tag{14}$$

pushing from outside against the surface.

The fluid velocity $\boldsymbol{u}$ is described by the Navier–Stokes equation (Chandrasekhar 1961)

$$\rho \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\epsilon \rho g \boldsymbol{e}_z + \nabla \cdot \boldsymbol{T} - \nabla p \tag{15}$$

where p is the pressure, ρ is the density and g is the constant gravitational acceleration. The hydrodynamic stress tensor for an incompressible fluid is defined as

$$T_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \tag{16}$$

with the viscosity μ. The continuity equation is

$$\nabla \cdot \boldsymbol{u} = 0. \tag{17}$$

The boundary conditions on a free surface require that the normal and tangential forces acting on the droplet surface vanish. Besides the hydrodynamical forces, which are expressed through the hydrodynamic stress tensor, we must consider the surface tension γ, gravity and the magnetic pressure. The stress $\boldsymbol{t}\delta S$ exerted on a surface element δS with unit normal $\boldsymbol{n}$ is expressed through the stress tensor (Acheson 1990)

$$\boldsymbol{t} = \boldsymbol{T} \cdot \boldsymbol{n} - p\boldsymbol{n}. \tag{18}$$

Because the surface tension force and the magnetic pressure are directed normal to the surface we obtain for the normal component of the boundary condition

$$- \left. \boldsymbol{t} \cdot \boldsymbol{n} \right|_{F=0} = p_{MAG} + \gamma \nabla \cdot \boldsymbol{n}. \tag{19}$$

The boundary conditions on the tangential components of the stress vector are

$$\boldsymbol{t}_\Theta \cdot \boldsymbol{t} = 0, \qquad \boldsymbol{t}_\varphi \cdot \boldsymbol{t} = 0, \tag{20}$$

with the tangential vectors defined as

$$\boldsymbol{t}_\Theta = \frac{\tau_\Theta}{|\tau_\Theta|}, \qquad \boldsymbol{t}_\varphi = \frac{\tau_\varphi}{|\tau_\varphi|} \tag{21}$$

with

$$\tau_\Theta = \frac{\partial}{\partial \Theta} r\boldsymbol{e}_r|_{F=0}, \qquad \tau_\varphi = \frac{\partial}{\partial \varphi} r\boldsymbol{e}_r|_{F=0}. \tag{22}$$

When the droplet is oscillating the velocity field must satisfy the kinematic boundary condition, which can be derived from (Acheson 1990)

$$\left. \frac{\mathrm{d}F}{\mathrm{d}t} \right|_{F=0} = 0. \tag{23}$$

From the above equations the pressure, the velocity field and the static deformation can be derived. One cannot derive an equation of motion for the surface in the viscous case. Instead we proceed similar to Chandrasekhar (1961) by assuming an exponential time dependence $\sim \exp(-\sigma t)$ for the motion and obtain a condition on σ.

3. Magnetic field, magnetic pressure and hydrostatic equilibrium

For completeness and easy reference, we briefly summarize the results for the magnetic field, magnetic pressure, hydrostatic pressure and static deformation. These quantities are not affected by the viscosity and hence are the same as in the work of Cummings & Blackburn (1991).

From equations (8) and (11) it follows that the magnetic field can be derived from a scalar magnetic potential ϕ, which satisfies the Laplace equation:

$$\boldsymbol{B} = \nabla\phi, \qquad \Delta\phi = 0 \tag{24}$$

Hence we make the following ansatz for the magnetic potential:

$$\phi = \left(H_{lm} \frac{r^l}{a^{l-1}} + I_{lm} \frac{a^{l+2}}{r^{l+1}} \right) Y_l^m(\Omega). \tag{25}$$

The coefficients H_{lm} describe the external magnetic field, the coefficients I_{lm} the induced magnetic field. For the spherical harmonics we use the definition given in Butkov (1968). As the induced field depends on the shape of the droplet it will change during the course of an oscillation. For small deformations the induced field may be expanded in powers of the deformation. The dynamic deformation is an infinitesimal quantity. It is not proportional to the expansion parameter ϵ, but an artifically introduced infinitesimal amplitude, as usual in the stability analysis of hydrodynamics (Chandrasekhar 1961). The static deformation is proportional to the perturbation parameter ϵ because it is generated by the external forces. To first order in the deformation, static or dynamic, we can write

$$I_{lm} = I_{lm}^{(0)} + I_{lm}^{(1)}, \tag{26}$$

where $I_{lm}^{(0)}$ is the induced magnetic field for a spherical droplet and $I_{lm}^{(1)}$ the magnetic field induced by the deformation. The induced magnetic field follows from the boundary condition (12). The surface normal $\boldsymbol{n}$ is to first order in the deformation

$$\boldsymbol{n} = \boldsymbol{e}_r - \boldsymbol{e}_\Theta \left(\xi_{lm} + R_{lm} \right) \frac{\partial Y_l^m}{\partial \Theta} - \boldsymbol{e}_\varphi \frac{\xi_{lm} + R_{lm}}{\sin(\Theta)} \frac{\partial Y_l^m}{\partial \varphi}. \tag{27}$$

From this equation one obtains (Cummings & Blackburn 1991) for the expansion coefficients of the induced magnetic field

$$I_{lm}^{(0)} = \frac{l}{l+1} H_{lm}, \tag{28}$$

for the zeroth order, and

$$I_{de}^{(1)} = H_{lm} \left(\xi_{uv} + R_{uv} \right) \langle de|uv|lm \rangle \frac{2l+1}{2(l+1)(d+1)} \left(d(d+1) + l(l+1) - u(u+1) \right) \tag{29}$$

for the first order. The expression in angular brackets stands for the integral

$$\langle de|lm|uv \rangle = \int \mathrm{d}\Omega \; Y_d^e(\Omega) \; Y_l^m(\Omega) \; Y_u^v(\Omega) \tag{30}$$

over a triple product of spherical harmonics. The arguments of the angular brackets must be considered as indices that obey the summation convention.

After the magnetic field is calculated to first order in the surface deformation, we now must look for the magnetic pressure. It was expanded in spherical harmonics and one obtains for the zeroth order in the deformation

$$p^{(0)}_{MAG,de} = \epsilon \frac{(2l+1)(2u+1)}{4\mu_0(l+1)(u+1)} H_{lm} H_{uv} \langle de|uv|lm\rangle \left(l(l+1) + u(u+1) - d(d+1)\right), \quad (31)$$

and

$$p^{(1)}_{MAG,lm} = m_{lm,uv}\left(\xi_{uv} + R_{uv}\right), \quad (32)$$

with the matrix

$$\begin{aligned} m_{rs,xz} = \epsilon & \frac{(2l+1)(2u+1)}{(u+1)(l+1)2\mu_0} H_{lm} H_{uv} \\ & \times \left[\langle rs|xz|de\rangle\langle de|uv|lm\rangle \left(d(d+1) - u(u+1) - l(l+1)\right)\right. \\ & + \langle rs|lm|de\rangle\langle de|xz|uv\rangle \left(l(l+1) + d(d+1) - r(r+1)\right) \\ & \left.\times (2d+2)^{-1}\left(d(d+1) + u(u+1) - x(x+1)\right)\right] \end{aligned} \quad (33)$$

for the first-order magnetic pressure expansion coefficients. The total magnetic pressure is given by the sum of (31) and (32).

The expression for the magnetic pressure must now be inserted into the hydrostatic pressure equilibrium to find the static equilibrium form of the droplet. When the velocity field is zero everywhere, the hydrodynamic flow equation simplifies to a time-independent equation for the static pressure p^S:

$$0 = -\nabla p^S - \epsilon \rho g \boldsymbol{e}_z. \quad (34)$$

From this the static pressure inside the droplet is

$$p^S = p_0 - \epsilon \rho g r \cos(\Theta). \quad (35)$$

The surface tension force on the deformed surface is to first order in the static deformation R_{lm}

$$\gamma \nabla \cdot \boldsymbol{n} = \frac{2\gamma}{a} + \frac{\gamma}{a} R_{lm} Y_l^m \left(l(l+1) - 2\right). \quad (36)$$

Expanding the static pressure (35) on the deformed surface to first order in the static deformation and making use of the orthogonality of the spherical harmonics, we obtain from equation (19)

$$\begin{aligned} (4\pi)^{1/2} p_0 \delta_{rs,00} - (4\pi)^{1/2} \frac{2\gamma}{a} \delta_{rs,00} = & \epsilon \rho g a (4\pi/3)^{1/2} \delta_{rs,10} \\ & + p^{(0)}_{MAG,rs} + \frac{\gamma}{a} \delta_{rs,xz} \left(r(r+1) - 2\right) R_{xz} + M_{rs,xz} R_{xz}, \end{aligned} \quad (37)$$

with

$$M_{rs,xz} = \epsilon \rho g a (4\pi/3)^{1/2} \langle rs|xz|10\rangle + m_{rs,xz}, \quad (38)$$

$$\delta_{ij,lk} = \delta_{i,l}\delta_{jk}. \quad (39)$$

This pressure equilibrium consists of three parts. The left-hand side of equation (37) represents the situation without external forces ($g = 0, \boldsymbol{B} = 0$). In the absence of external forces it is clear that the equilibrium shape of the droplet is spherical and the pressure is given by

$$p_0 = \gamma \frac{2}{a}. \quad (40)$$

In the presence of gravity the magnetic pressure must compensate the gravity force. The zeroth-order magnetic pressure $p^{(0)}_{MAG}$ and the pressure contribution from gravity depend on the angles Θ and φ differently in general. Hence equilibrium can be established only by a deformation. To first order in ϵ one obtains

$$R_{lm} = \frac{\epsilon \rho g a (4\pi/3)^{1/2} \delta_{lm,10} + p^{(0)}_{MAG,lm}}{(\gamma/a)(l(l+1)-2)}; \tag{41}$$

equation (41) guarantees only hydrostatic equilibrium. Mechanical equilibrium can only be achieved if the magnetic field compensates the gravitational field and provides a restoring force for deviations of the droplet's centre of mass from its equilibrium position. This puts a condition on the coefficients H_{lm} contained in equation (41) via the magnetic pressure term. In the case of a linear magnetic field, this condition is given explicitly in equation (87).

4. Fluid flow

In the case of an inviscid liquid, it was not necesarry to consider the fluid flow explicitly (Cummings & Blackburn 1991). In the present case of a viscous liquid, the boundary conditions require an investigation of the fluid flow.

4.1. *Navier–Stokes equation*

The flow field is calculated from the Navier–Stokes equation. The linearized Navier Stokes equation for an incompressible liquid in a gravity field reads

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p + \mu \Delta \boldsymbol{u} - \epsilon \rho g \boldsymbol{e}_z. \tag{42}$$

It follows from the incompressibility that the velocity field $\boldsymbol{u}$ can be written as a sum of a poloidal $\boldsymbol{S}$ and a toroidal field $\boldsymbol{Q}$, as defined by Chandrasekhar (1961):

$$\boldsymbol{u} = \boldsymbol{S} + \boldsymbol{Q}. \tag{43}$$

The definition of the poloidal and toroidal fields is

$$\boldsymbol{S} = \nabla \times \nabla \times \frac{\boldsymbol{r}}{r} S(\boldsymbol{r}, t), \qquad \boldsymbol{Q} = \nabla \times \frac{\boldsymbol{r}}{r} Q(\boldsymbol{r}, t). \tag{44}$$

For the potentials $S(\boldsymbol{r}, t)$ and $Q(\boldsymbol{r}, t)$ we perform a separation ansatz

$$S(\boldsymbol{r}, t) = S_{lm}(r, t) Y_l^m(\Omega), \qquad Q(\boldsymbol{r}, t) = Q_{lm}(r, t) Y_l^m(\Omega). \tag{45}$$

To obtain a solution for the poloidal and the toroidal potentials we assume an exponential time dependence $S_{lm} \sim \mathrm{e}^{-\sigma t}$ and $Q_{lm} \sim \mathrm{e}^{-\sigma t}$ of the potentials with complex σ. The pressure satisfies the Laplace equation and may be expanded as

$$p = \alpha_{lm} r^l Y_l^m, \tag{46}$$

with expansion coefficients α_{lm}. From the Navier–Stokes equation, the exponential time dependence and the expansions in spherical harmonics for the poloidal and toroidal potential and the pressure, we obtain for the potentials

$$S_{lm}(x) = s_{lm} x\, c\, j_l(x) + \frac{\alpha_{lm}}{\rho \sigma (l+1)} x^{l+1} c^{l+1} \tag{47}$$

and

$$Q_{lm}(x) = q_{lm} x\, c\, j_l(x), \tag{48}$$

where $x = r/c$, $c = [\mu/(\sigma\rho)]^{1/2}$ and s_{lm} and q_{lm} are expansion coefficients to be determined from the boundary conditions. The j_l are the spherical Bessel functions of order l. No summation over l is performed in equations (47) and (48).

The flow field components are explicitly

$$u_r = \frac{l(l+1)}{r}\left(s_{lm}j_l + \frac{\alpha_{lm}}{\rho\sigma(l+1)}r^l\right)Y_l^m, \tag{49}$$

$$u_\Theta = \left(\frac{l+1}{r}s_{lm}j_l - s_{lm}\frac{j_{l+1}}{c} + \frac{\alpha_{lm}}{\rho\sigma}r^{l-1}\right)\partial_\Theta Y_l^m + \frac{q_{lm}j_l}{\sin(\Theta)}\partial_\varphi Y_l^m, \tag{50}$$

$$u_\varphi = \frac{1}{\sin(\Theta)}\left(\frac{l+1}{r}s_{lm}j_l - s_{lm}\frac{j_{l+1}}{c} + \frac{\alpha_{lm}}{\rho\sigma}r^{l-1}\right)\partial_\varphi Y_l^m - q_{lm}j_l\partial_\Theta Y_l^m. \tag{51}$$

The functional dependence of the j_l and the Y_l^m is omitted for brevity.

4.2. *The boundary conditions on the tangential stresses*

For evaluation of the boundary conditions (20) the surface tangential vectors are required. They are

$$\boldsymbol{t}_\Theta = (\xi_{lm} + R_{lm})\partial_\Theta Y_l^m \boldsymbol{e}_r + \boldsymbol{e}_\Theta \tag{52}$$

and

$$\boldsymbol{t}_\varphi = \frac{\xi_{lm} + R_{lm}}{\sin(\Theta)}\partial_\varphi Y_l^m \boldsymbol{e}_r + \boldsymbol{e}_\varphi. \tag{53}$$

Then we obtain from equations (18), (20), (27), (52) and (53) for the boundary conditions up to first order in ϵ

$$\boldsymbol{t}_\Theta \cdot \boldsymbol{T} \cdot \boldsymbol{n} = T_{\Theta r} - R_{lm}\left[T_{\Theta\Theta}\partial_\Theta Y_l^m + T_{\Theta\varphi}\frac{\partial_\varphi Y_l^m}{\sin(\Theta)}\right] + T_{rr}R_{lm}\partial_\Theta Y_l^m = 0 \tag{54}$$

and

$$\boldsymbol{t}_\varphi \cdot \boldsymbol{T} \cdot \boldsymbol{n} = T_{\varphi r} - R_{lm}\left[T_{\varphi\Theta}\partial_\Theta Y_l^m + T_{\varphi\varphi}\frac{\partial_\varphi Y_l^m}{\sin(\Theta)}\right] + T_{rr}R_{lm}\frac{\partial_\varphi Y_l^m}{\sin(\Theta)} = 0, \tag{55}$$

because the static deformation is of first order in ϵ. $T_{\alpha,\beta}$ are the components of the stress tensor in spherical coordinates. No summation is performed over the indices α, β. The summation convention will not be applied if the indices denote spherical coordinates.

By use of these equations the expansion coefficient s_{lm} can be expressed through the α_{lm}. These coefficients are dynamical quantities and hence are proportional to the dynamic deformation ξ_{lm}. The relations between these dynamical quantities are required to first order in ϵ. We therefore write

$$s_{lm} = s_{lm}^{(0)} + \epsilon s_{lm}^{(1)} \tag{56}$$

and, accordingly, for the other coefficients

$$q_{lm} = q_{lm}^{(0)} + \epsilon q_{lm}^{(1)}, \tag{57}$$

$$\alpha_{lm} = \alpha_{lm}^{(0)} + \epsilon \alpha_{lm}^{(1)}. \tag{58}$$

With this we obtain for the zeroth order

$$s_{lm}^{(0)} = c_{s\alpha}(l)\frac{\alpha_{lm}^{(0)}}{\sigma\rho}a^{l-2} \tag{59}$$

with

$$c_{s\alpha}(l) = -\left(2a^2c^2(l-1)\right)\left[2(l^2-1)c^2 j_l(a/c) - (2l+1)a\,c\,j_{l+1}(a/c) + a^2 j_{l+2}(a/c)\right]^{-1}. \tag{60}$$

For the first order we obtain

$$s_{lm}^{(1)} = -c\left(c_{\Theta r}(l,a/c)l(l^2-1)\right)^{-1}\left(2a^{-2+l}c\,\alpha_{lm}^{(1)}l^2(l+1) - \frac{R_{uv}}{\epsilon}c\,\alpha_{ik}^{(0)}a^{i-2}\Sigma_{lm,uv,ik}\right). \tag{61}$$

The coefficients $\Sigma_{lm,uv,ik}$ and $c_{\Theta r}$ are defined in the Appendix.

For the toroidal field we have $q_{lm}^{(0)} = 0$. Because the zeroth order of the toroidal potential vanishes and the radial component of the toroidal field vanishes the contributions to the pressure equilibrium and kinematic boundary condition are of order ϵ^2. Hence the toroidal field does not affect the oscillation spectrum at the perturbation order considered.

5. Pressure equilibrium

From the pressure equilibrium (19) we obtain a relation between the coefficients $\alpha_{lm}^{(0)}$ and $\alpha_{lm}^{(1)}$ and the dynamic deformation ξ_{lm}. The pressure equilibrium is

$$\tilde{p}|_{F=0} = \gamma\nabla\cdot\boldsymbol{n}|_{F=0} + p_{ext} + \boldsymbol{n}\cdot\boldsymbol{T}\cdot\boldsymbol{n}. \tag{62}$$

The external pressure to first order in ϵ is $p_{ext} = \epsilon g\rho a\cos(\Theta) + p_{MAG}^{(0)} + Y_l^m M_{lm,uv}\xi_{uv}$. It is generated by gravity and the Lorentz force, with the matrix M defined in equation (38). The pressure $\tilde{p}$ is defined as

$$\tilde{p} = p + \epsilon g\rho r\cos(\Theta), \tag{63}$$

so that the influence of the external forces is completely contained in p_{ext}. On the surface $\tilde{p}$ is to first order in ϵ

$$\tilde{p}|_{F=0} = p_0 + \left(\alpha_{lm}^{(0)} + \epsilon\alpha_{lm}^{(1)}\right)a^l Y_l^m + \alpha_{ik}^{(0)} i a^i R_{uv} Y_l^m\langle lm|ik|uv\rangle. \tag{64}$$

The dynamic part of the surface tension term is to first order in ϵ

$$\gamma\nabla\cdot\boldsymbol{n} = \frac{\gamma}{a}(l(l+1)-2)\xi_{lm}Y_l^m + \frac{\gamma}{a}R_{uv}\xi_{ik}(4-2i(i+1)-2u(u+1))Y_l^m\langle lm|ik|uv\rangle. \tag{65}$$

The contribution from the normal component of the stress tensor is

$$\boldsymbol{n}\cdot\boldsymbol{T}\cdot\boldsymbol{n}|_{F=0} = T_{rr}|_{r=a} + T'_{rr}|_{r=a}R_{uv}Y_u^v - 2R_{uv}\left[T_{r\Theta}\partial_\Theta Y_u^v + T_{r\varphi}\frac{\partial_\varphi Y_u^v}{\sin(\Theta)}\right] \tag{66}$$

$$= T_{rr}|_{r=a} + T'_{rr}|_{r=a}R_{uv}Y_u^v, \tag{67}$$

because the zeroth order of the stress tensor components $T_{r\Theta}$ and $T_{r\varphi}$ must vanish on $r = a$, if the boundary condition is to be satisfied by each order in ϵ separately. T'_{rr} is the derivative of T_{rr} with respect to r. An explicit expression for the normal component of the stress vector can be derived from the expressions for $s_{lm}^{(0)}$ and $s_{lm}^{(1)}$. For the zeroth order we obtain

$$\alpha_{lm}^{(0)} = \gamma\xi_{lm}\frac{(l^2+l-2)a^{-l+1}}{a^2 - c^2 D_{rr}(l,a/c)}, \tag{68}$$

and for the first order

$$\alpha^{(1)}_{lm} = \left(\left[-2l(l-1)c^2 + a^2\right] c_{\Theta r}(l,a/c) + 2(l-1)c^2 c_{rr}(l,a/c)\right)^{-1} \times \left\{ -c_{\Theta r}(l,a/c)\frac{R_{uv}}{\epsilon}\frac{\gamma \xi_{ik}}{a^{l-1}}(-4 + 2u(u+1) + 2i(i+1)) + \alpha^{(0)}_{ik}\frac{R_{uv}}{\epsilon}\Xi_{lm,uv,ik} + c_{\Theta r}(l,a/c)a^{2-l}\frac{M_{lm,ik}}{\epsilon}\xi_{ik} \right\}. \tag{69}$$

The coefficients $c_{\Theta r}, c_{rr}$ and $\Xi_{lm,uv,ik}$ are defined in the Appendix.

6. The eigenvalue equation for the surface oscillations

Finally the kinematic boundary condition is used for the derivation of the eigenvalue equation for σ. From equation (23) it follows that

$$-\sigma a \xi_{lm} Y_l^m = \boldsymbol{u} \cdot \boldsymbol{n}. \tag{70}$$

The surface normal component of the velocity is

$$\boldsymbol{u} \cdot \boldsymbol{n} = u_r|_{F=0} - u_\Theta R_{uv} \partial_\Theta Y_u^v - u_\varphi R_{uv} \frac{\partial_\varphi Y_u^v}{\sin(\Theta)}, \tag{71}$$

with

$$u_r|_{F=0} = u_r|_{r=a} + \partial_r u_r|_{r=a} a R_{uv} Y_u^v. \tag{72}$$

For the zeroth order in ϵ we obtain for the normal component of the velocity

$$(\boldsymbol{u} \cdot \boldsymbol{n})^{(0)} = -Y_l^m \alpha^{(0)}_{lm} \frac{l\left(c(2l+1)a^l Q_1(l,a/c) - a^{l+1}Q_2(l,a/c)\right)}{\left(2c^2(l^2-1) - (2l+1)acQ_1(l,a/c) + a^2 Q_2(l,a/c)\right)\sigma\rho}. \tag{73}$$

Here we used the abbreviation $Q_i(l,x)$ for the ratio of the different spherical Bessel functions:

$$Q_i(l,x) = \frac{j_{l+i}(x)}{j_l(x)}. \tag{74}$$

For the first order in ϵ we obtain for the normal component of the velocity

$$(\boldsymbol{u} \cdot \boldsymbol{n})^{(1)} = -\frac{\left(-a^2 c_{\Theta r}(l,a/c) + 2c^2(l^2-1)j_l(a/c)\rho\sigma\right) a^{l-2} l}{a c_{\Theta r}(l,a/c)\rho\sigma} \epsilon \alpha^{(1)}_{lm} Y_l^m + R_{uv} \alpha^{(0)}_{ik} Y_l^m \Lambda_{lm,uv,ik}. \tag{75}$$

The coefficient $\Lambda_{lm,uv,ik}$ is given in the Appendix.

Inserting the expressions for the normal component of the velocity (73) and (75) into the kinematic boundary condition yields the required equation for the eigenmodes σ.

The equation for the zeroth order (73) alone gives the familiar eigenvalue equation derived by Chandrasekhar (1961). This can be seen directly, if another recurrence relation for the spherical Bessel functions

$$j_{l+1}(x) + j_{l-1}(x) = \frac{2l+1}{x} j_l(x) \tag{76}$$

is inserted into (73) and this into (70). This yields equation (2).

The other known limiting case is the infinite Reynolds number limit. This is the approximation of an inviscid liquid as calculated by Cummings and Blackburn (1991).

In the following section we extend their treatment by considering the high Reynolds number region.

7. High Reynolds numbers

The kinematic boundary condition cannot be solved analytically without further approximation. We investigate the case of high Reynolds numbers, which means that the viscosity is small. For high Reynolds numbers $c/a = (\mu/a^2\sigma\rho)^{1/2}$ becomes very small. Therefore the argument $x = r/c$ of the Bessel functions becomes large and we need the asymptotic expansions for the Bessel functions. We evalute all contributions to first non-vanishing order in c/a. For the asymptotic expansion of the zeroth-order normal component of the velocity (73), we obtain

$$(\boldsymbol{u}\cdot\boldsymbol{n})^{(0)} = \frac{\gamma l(l+2)(l-1)}{a^2\sigma\rho}\left(1+\frac{2(2l+1)(l-1)\mu}{a^2\sigma\rho}\right)\xi_{lm}Y_l^m. \tag{77}$$

Because we calculate everything to first order in ϵ it follows from perturbation theory (Fetter & Walecka 1980) that only the diagonal elements of the kinematic boundary condition, which is a matrix equation, are required. From perturbation theory it follows that, for the first-order correction to the eigenvalues of the matrix considered, only the diagonal elements of the perturbation matrix in the unperturbed eigenvector system are required. This means substituting i,k by l,m in equation (75). For the asymptotic expansion of the first-order normal component of the velocity the result is

$$\begin{aligned}
(\boldsymbol{u}\cdot\boldsymbol{n})^{(1)} \\
&= l\frac{\sigma\rho a^2 + 2\mu(2l+1)(l-1)}{\sigma^2\rho^2 a^3}M_{lm,lm}\xi_{lm}Y_l^m \\
&-\gamma\frac{6l^3+5lu^2+5lu+u^2l^2+ul^2-2u^2-2u+6l^2-12l}{2a^2\sigma\rho}\langle lm|uv|lm\rangle\xi_{lm}R_{uv}Y_l^m \\
&-\frac{\mu\gamma(l-1)}{a^4\sigma^2\rho^2}\langle lm|uv|lm\rangle\xi_{lm}R_{uv}Y_l^m \\
&\times\left(20l^4+30l^3+3l^3u^2+3l^3u-30l^2+11u^2l^2+11ul^2-lu^2-lu-20l+2u^2+2u\right) \\
&+\frac{2\gamma(l+2)(l-1)l\mu}{a^4\sigma^2\rho^2}\xi_{lm}R_{uv}\{lm|uv|lm\}Y_l^m.
\end{aligned} \tag{78}$$

No summation over l is performed in equation (78). The coeffcients $\{lm|uv|ik\}$ are given in the Appendix.

From equations (70), (77) and (78) the eigenvalues σ may be calculated. As we calculated everything to first order in the external forces and the viscosity, the eigenvalues σ must be expanded similarly:

$$\sigma = \mathrm{i}\omega^{(0)} + \Gamma^{(0)} + \mathrm{i}\epsilon\omega^{(1)} + \epsilon\Gamma^{(1)}. \tag{79}$$

For zeroth order in the external forces we obtain

$$\omega^{(0)} = \pm\left(\frac{\gamma l(l+2)(l-1)}{a^3\rho}\right)^{1/2} \tag{80}$$

for the frequencies and

$$\Gamma^{(0)} = \frac{(2l+1)(l-1)}{\rho a^2} \tag{81}$$

for the damping constants. This is the result of Chandrasekhar (1961).

l,m	u,v	$\{lm\|uv\|lm\}$	$\langle lm\|uv\|lm\rangle$
$1,\pm1$	$2,0$	$-\frac{3}{10}(5/\pi)^{1/2}$	$-\frac{1}{10}(5/\pi)^{1/2}$
$1,0$	$2,0$	$\frac{3}{5}(5/\pi)^{1/2}$	$\frac{1}{5}(5/\pi)^{1/2}$
$2,\pm2$	$2,0$	$\frac{99}{28}(5/\pi)^{1/2}$	$-\frac{1}{7}(5/\pi)^{1/2}$
$2,\pm1$	$2,0$	$-\frac{81}{28}(5/\pi)^{1/2}$	$\frac{1}{14}(5/\pi)^{1/2}$
$2,0$	$2,0$	$-\frac{9}{7}(5/\pi)^{1/2}$	$\frac{1}{7}(5/\pi)^{1/2}$
$1,\pm1$	$3,0$	0	0
$1,0$	$3,0$	0	0
$2,\pm2$	$3,0$	0	0
$2,\pm1$	$3,0$	0	0
$2,0$	$3,0$	0	0
$1,\pm1$	$4,0$	0	0
$1,0$	$4,0$	0	0
$2,\pm2$	$4,0$	$\frac{395}{28}\pi^{-1/2}$	$\frac{1}{14}\pi^{-1/2}$
$2,\pm1$	$4,0$	$-\frac{635}{28}\pi^{-1/2}$	$-\frac{2}{7}\pi^{-1/2}$
$2,0$	$4,0$	$-\frac{120}{7}\pi^{-1/2}$	$\frac{3}{7}\pi^{-1/2}$

TABLE 1. List of coefficients $\{lm|uv|lm\}$ defined in equation (A3), and $\langle lm|uv|lm\rangle$ defined in equation (30).

The first orders in the external forces $\omega^{(1)}$ and $\Gamma^{(1)}$ are obtained by substituting these results into equation (78). Instead of calculating $\omega^{(1)}$, it is convenient to calculate the square $\left(\omega^{(0)}+\epsilon\omega^{(1)}\right)^2$. This follows from the dependence of (77), (78) and the kinematic boundary condition (70) on the eigenvalue σ. If the kinematic boundary condition is multiplied with σ one obtains

$$\left(\omega^{(0)}+\epsilon\omega^{(1)}\right)^2=\gamma\frac{l(l+2)(l-1)}{a^3\rho}+\frac{l}{a^2\rho}M_{lm,lm}$$
$$-\frac{1}{2a^3\rho}\gamma\left(6l^3+5lu^2+5lu+u^2l^2+ul^2-2u^2-2u+6l^2-12l\right)R_{uv}\langle lm|uv|lm\rangle. \tag{82}$$

From the contribution to the kinematic boundary condition, which is proportional to μ, we obtain for the damping

$$\Gamma^{(0)}+\epsilon\Gamma^{(1)}=\frac{\mu}{a^2\rho}(2l+1)(l-1)+\frac{\mu}{a^2\rho}R_{uv}\{lm|uv|lm\}$$
$$-\frac{\mu}{2a^2\rho l}\left(8l^3+l^2u^2+l^2u-4l^2-2lu-4l-2lu^2+2u+2u^2\right)R_{uv}\langle lm|uv|lm\rangle. \tag{83}$$

These results are of first order in the expansion paprameter ϵ via the static deformation R_{uv} and the matrix elements $M_{lm,lm}$. Equations (83) and (82) represent the general solution of the problem, correct to first order in the perturbation parameter ϵ, which will be set to one in the following.

For the frequency, we obtain (equation (82)) the same result as Cummings & Blackburn (1991). This implies that there is no correction to their result to first order in viscosity, just as in the absence of external forces (Reid 1960). The damping due to viscosity of the oscillations of an aspherical droplet is given by equation (83). It has not been calculated before, and is the central result of our paper.

8. Explicit results for a linear magnetic field

The above general formulas will now be evaluated for the special case of a linear magnetic field. This means that only the two expansion coefficients H_{10} and H_{20} are non-zero.

The coefficients required for the evaluation of equation (83) for a linear magnetic field are listed in table 1.

The static deformation is (Cummings & Blackburn 1991)

$$\begin{aligned} R_{20} &= -\frac{a}{4\gamma}\left(\frac{5}{\pi}\right)^{1/2}\left(-\frac{9}{40}\frac{H_{10}^2}{\mu_0}+\frac{25}{42}\frac{H_{20}^2}{\mu_0}\right), \\ R_{30} &= \frac{3a}{140\gamma}\left(\frac{105}{\pi}\right)^{1/2}\frac{H_{10}H_{20}}{\mu_0}, \\ R_{40} &= \frac{25a}{189\gamma\pi^{1/2}}\frac{H_{20}^2}{\mu_0}. \end{aligned} \tag{84}$$

For the complex frequencies $\sigma = \mathrm{i}\omega + \Gamma$ we obtain

$$\left.\begin{aligned} \omega_{1,\pm1}^2 &= \frac{15}{8}\frac{H_{20}^2}{\mu_0 a^2\rho\pi}, \quad \omega_{1,0}^2 = \frac{15}{2}\frac{H_{20}^2}{\mu_0 a^2\rho\pi}, \\ \omega_{2,\pm2}^2 &= \frac{8\gamma}{a^3\rho} - \frac{1025}{294}\frac{H_{20}^2}{\mu_0 a^2\rho\pi} + \frac{81}{56}\frac{H_{10}^2}{\mu_0 a^2\rho\pi}, \\ \omega_{2,\pm1}^2 &= \frac{8\gamma}{a^3\rho} + \frac{2775}{196}\frac{H_{20}^2}{\mu_0 a^2\rho\pi} + \frac{81}{280}\frac{H_{10}^2}{\mu_0 a^2\rho\pi}, \\ \omega_{2,0}^2 &= \frac{8\gamma}{a^3\rho} + \frac{8450}{588}\frac{H_{20}^2}{\mu_0 a^2\rho\pi} - \frac{27}{280}\frac{H_{10}^2}{\mu_0 a^2\rho\pi}, \end{aligned}\right\} \tag{85}$$

and

$$\left.\begin{aligned} \Gamma_{1,\pm1} &= \Gamma_{1,0} = 0, \\ \Gamma_{2,\pm2} &= \frac{\mu}{a^2\rho}\left(5+\frac{1359}{896}\frac{H_{10}^2 a}{\mu_0\pi\gamma} - \frac{14125}{6048}\frac{H_{20}^2 a}{\mu_0\pi\gamma}\right), \\ \Gamma_{2,\pm1} &= \frac{\mu}{a^2\rho}\left(5-\frac{963}{896}\frac{H_{10}^2 a}{\mu_0\pi\gamma} + \frac{3625}{6048}\frac{H_{20}^2 a}{\mu_0\pi\gamma}\right), \\ \Gamma_{2,0} &= \frac{\mu}{a^2\rho}\left(5-\frac{99}{112}\frac{H_{10}^2 a}{\mu_0\pi\gamma} + \frac{125}{36}\frac{H_{20}^2 a}{\mu_0\pi\gamma}\right). \end{aligned}\right\} \tag{86}$$

As mentioned above in §3, for levitation the Lorentz force must compensate gravity. The force balance requires $F_L + F_g = 0$, from which follows

$$\sqrt{15}a^2\frac{H_{10}H_{20}}{\mu_0} = -\frac{4\pi}{3}a^3\rho g, \tag{87}$$

as was derived by Cummings and Blackburn (1991, equation (5.18)). The levitation is indeed linearly stable, because the frequencies of the $l = 1$ modes, which describe the oscillation of the centre of mass (equation (85)), are all real. We refer the interested reader to the paper of Cummings & Blackburn for details on the location of the droplet in the magnetic field.

Frequencies and damping of the $l = 2$ modes can then be expressed through an average of the squares of the translational frequencies $\overline{\omega_1^2}$ and the Rayleigh frequency ω_R. With

$$\overline{\omega_1^2} = \tfrac{1}{3}\left(2\omega_{1,\pm1}^2 + \omega_{10}^2\right). \tag{88}$$

and

$$\omega_R^2 = \frac{8\gamma}{\rho a^3} \tag{89}$$

we obtain (Cummings & Blackburn 1991), using equation (87)

$$\left.\begin{aligned} \omega_{2,\pm 2}^2 &= \omega_R^2 - \frac{410}{441}\overline{\omega_1^2} + \frac{9}{14}\frac{g^2}{a^2\overline{\omega_1^2}}, \\ \omega_{2,\pm 1}^2 &= \omega_R^2 + \frac{185}{49}\overline{\omega_1^2} + \frac{9}{70}\frac{g^2}{a^2\overline{\omega_1^2}}, \\ \omega_{2,0}^2 &= \omega_R^2 + \frac{1690}{441}\overline{\omega_1^2} - \frac{3}{70}\frac{g^2}{a^2\overline{\omega_1^2}}, \end{aligned}\right\} \tag{90}$$

and hence for the average of the squares of the frequencies of the $l = 2$ modes

$$\overline{\omega_2^2} = \omega_R^2 + \frac{40}{21}\overline{\omega_1^2} + \frac{3}{10}\frac{g^2}{a^2\overline{\omega_1^2}}. \tag{91}$$

For the damping the results are

$$\left.\begin{aligned} \Gamma_{2,0} &= \Gamma_C + \frac{\mu}{\rho a^2}\left(\frac{200}{27}\frac{\overline{\omega_1^2}}{\omega_R^2} - \frac{22}{7}\frac{g^2}{a^2\overline{\omega_1^2}\omega_R^2}\right), \\ \Gamma_{2,\pm 1} &= \Gamma_C + \frac{\mu}{\rho a^2}\left(\frac{725}{567}\frac{\overline{\omega_1^2}}{\omega_R^2} - \frac{107}{28}\frac{g^2}{a^2\overline{\omega_1^2}\omega_R^2}\right), \\ \Gamma_{2,\pm 2} &= \Gamma_C + \frac{\mu}{\rho a^2}\left(-\frac{2825}{567}\frac{\overline{\omega_1^2}}{\omega_R^2} + \frac{151}{28}\frac{g^2}{a^2\overline{\omega_1^2}\omega_R^2}\right), \end{aligned}\right\} \tag{92}$$

and the average of the damping constants is

$$\overline{\Gamma}_2 = \Gamma_C. \tag{93}$$

On average, the external forces have no influence on the damping of the surface waves.

In figures 1 and 2 the relative changes of the frequency and damping of the $l = 2$ modes are shown as functions of the the average translational frequency $\overline{\nu}_1 = [\overline{\omega_1^2}/(4\pi^2)]^{1/2}$. As parameters we choose a radius $a = 3$ mm and a Rayleigh frequency $\nu_R = \omega_R/(2\pi) = 40$ Hz. The relative change is about a few percent for $\overline{\nu}_1 \approx 5$ Hz. Both for lower and higher translational frequencies, the influence of the external forces on the frequencies and damping increases. This is understood easily from formulas (90) and (92).

9. Discussion

We investigated the influence of gravity and Lorentz force on the damping of surface waves on electromagnetically levitated liquid-metal droplets. Formulas were derived which describe the dynamics for arbitrary values of the viscosity and arbitrary magnetic fields. The calculations were performed with the aid of the computer algebra program Maple (University of Waterloo, Canada, 1981). The practically important limiting case of high Reynolds numbers and a linear magnetic field was evaluated explicitly. A correction of the damping due to the static deformation was found. For

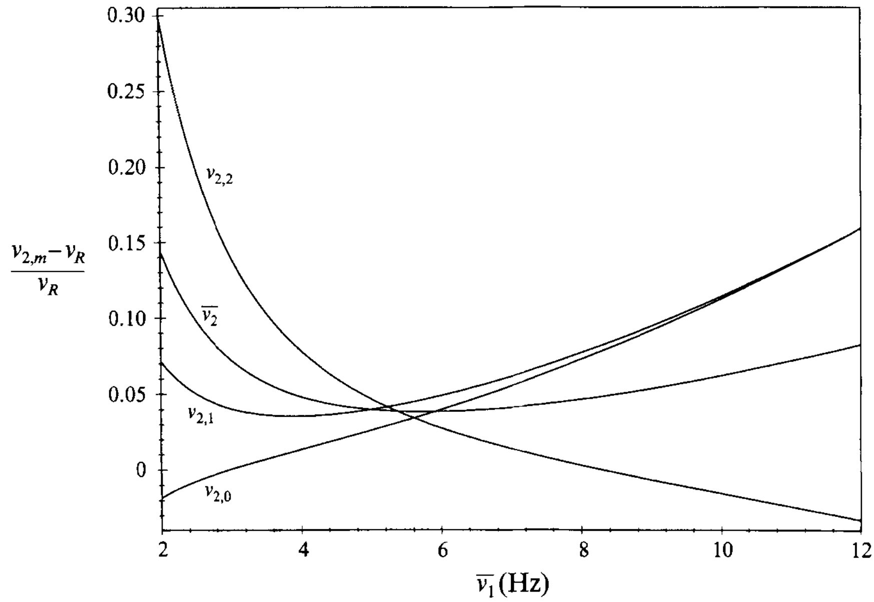

FIGURE 1. The relative change of the frequencies $\nu_{2,\pm m}$ and their average is shown as a function of the average translational frequency $\overline{\nu_1}$ for a radius of $a = 3\,\text{mm}$ and a Rayleigh frequency $\nu_R = [8\gamma/(4\pi^2\rho a^3)]^{1/2}$ of 40 Hz.

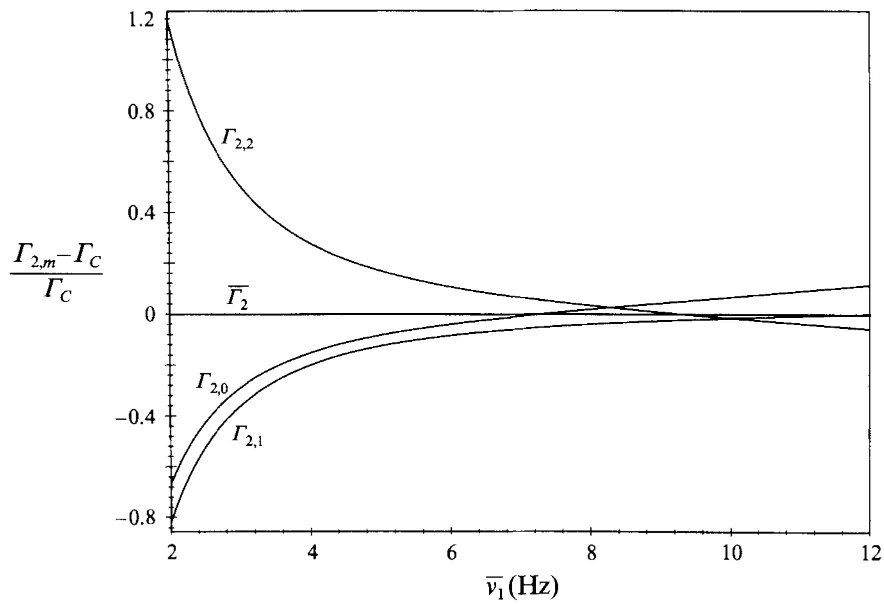

FIGURE 2. The relative change of the damping constants $\Gamma_{2,\pm m}$ is shown as a function of the average translational frequency $\overline{\nu_1}$ for a radius of $a = 3mm$ and a Rayleigh frequency $\nu_R = [8\gamma/(4\pi^2\rho a^3)]^{1/2}$ of 40 Hz.

fixed l-values the correction disappears if the average over the different m-values is considered. In contrast to this, the influence of the external forces on the frequencies does not disappear on average. For some typical values of surface tension γ and droplet radius a, a correction to the damping of the surface waves of a few percent is found for realistic values of the magnetic field, which was expressed through the average translational frequency $\overline{\nu_1}$.

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Appendix.

The coefficient $\sum_{lm,uv,ik}$ required in equation (61) for the flow field is given by

$$\sum\nolimits_{lm,uv,ik} = -D_{\Theta\Theta 1}(i, a/c)\langle\langle\langle lm|uv|ik\rangle\rangle\rangle + \frac{a}{c}\frac{\partial}{\partial x}D_{\Theta r}(i,x)\Big|_{x=a/c}\langle\langle\langle lm|ik|uv\rangle\rangle\rangle$$
$$+D_{rr}(i, a/c)\langle\langle\langle lm|uv|ik\rangle\rangle\rangle + D_{\Theta\Theta 2}(i, a/c)\{lm|uv|ik\}. \quad \text{(A 1)}$$

Here the following abbreviation was used:

$$\langle\langle\langle lm|uv|ik\rangle\rangle\rangle = \tfrac{1}{2}(l(l+1) + u(u+1) - i(i+1))\langle lm|uv|ik\rangle. \quad \text{(A 2)}$$

The coefficients $\{lm|uv|ik\}$ are given by

$$\{lm|uv|ik\} = \int \mathrm{d}\Omega\, Y_l^m$$
$$\times\left\{\frac{1}{\sin(\Theta)}\frac{\partial}{\partial\Theta}\sin(\Theta)\left(\left[\frac{\partial^2}{\partial\Theta^2}Y_i^k\right]\left[\frac{\partial}{\partial\Theta}Y_u^v\right]\right.\right.$$
$$\left.+\left[\frac{1}{\sin(\Theta)}\frac{\partial^2}{\partial\Theta\partial\varphi}Y_i^k - \frac{\cot(\Theta)}{\sin(\Theta)}\frac{\partial}{\partial\varphi}Y_i^k\right]\frac{1}{\sin(\Theta)}\left[\frac{\partial}{\partial\varphi}Y_u^v\right]\right)$$
$$+\frac{1}{\sin(\Theta)}\frac{\partial}{\partial\varphi}\left(\left[\frac{1}{\sin^2(\Theta)}\frac{\partial^2}{\partial\varphi^2}Y_i^k + cot(\Theta)\frac{\partial}{\partial\Theta}Y_i^k\right]\frac{1}{\sin(\Theta)}\left[\frac{\partial}{\partial\varphi}Y_u^v\right]\right.$$
$$\left.\left.+\left[\frac{1}{\sin(\Theta)}\frac{\partial^2}{\partial\Theta\partial\varphi}Y_i^k - \frac{\cot(\Theta)}{\sin(\Theta)}\frac{\partial}{\partial\varphi}Y_i^k\right]\left[\frac{\partial}{\partial\Theta}Y_u^v\right]\right)\right\}. \quad \text{(A 3)}$$

The coefficients $\{lm|uv|ik\}$ required for the evaluation of the case of a linear magnetic field are listed in table 1.

The coefficients D_{ij} are given by

$$D_{rr}(l,x) = -\frac{2l}{c^3x^3}\left((1-l)a^{-l+2}x^{l+1}c^{l+1}\right.$$
$$\left.+\,c_{s\alpha}(l)\left[xc(1-l^2)j_l(x) + (l+1)cx^2 j_{l+1}(x)\right]\right), \quad \text{(A 4)}$$

$$D_{\Theta\Theta 1}(l,x) = \frac{2l}{x^2c^2}\left(c_{s\alpha}(l)(l+1)j_l(x) + a^{-l+2}x^lc^l\right), \quad \text{(A 5)}$$

$$D_{\Theta\Theta 2}(l,x) = \frac{2}{c^2x^2}\left(c_{s\alpha}(l)[-(l+1)j_l(x) + xj_{l+1}(x) - a^{-l+2}x^lc^l\right), \quad \text{(A 6)}$$

$$D_{\Theta r}(l,x) = \frac{1}{c^3x^3}\left(2(l-1)a^{-l+2}x^{l+1}c^{l+1} + cc_{s\alpha}(l)\right.$$
$$\left.\times\left[2x(l^2-l)j_l(x) - x^2(2l+1)j_{l+1}(x) + x^3j_{l+2}(x)\right]\right), \quad \text{(A 7)}$$

and $c_{\Theta r}$ is

$$c_{\Theta r}(l,x) = \frac{1}{x^2}\rho\sigma(2(l^2-1)j_l(x) - x(2l+1)j_{l+1}(x) + x^2 j_{l+2}(x)). \tag{A 8}$$

For the first-order dynamic pressure we require in equation (69) the expression $\Xi_{lm,uv,ik}$. It is

$$\begin{aligned}\Xi_{lm,uv,ik} = &-c_{\Theta r}(l,a/c)a^{i-l+1}\left(ia - c\,\frac{\partial}{\partial x}D_{rr}(i,x)\bigg|_{x=a/c}\right)\langle lm|uv|ik\rangle.\\ &-c_{rr}(l,a/c)\frac{c}{l(l+1)a^{l-i-1}}\,\frac{\partial}{\partial x}D_{rr}(i,x)\bigg|_{x=a/c}\langle\langle\langle lm|ik|uv\rangle\rangle\rangle\\ &+c_{rr}(l,a/c)\frac{c^2}{l(l+1)a^{l-i}}[D_{\Theta\Theta 1}(i,a/c) - D_{rr}(i,a/c)]\langle\langle\langle lm|uv|ik\rangle\rangle\rangle\\ &-c_{rr}(l,a/c)\frac{c^2}{l(l+1)a^{l-i}}D_{\Theta\Theta 2}(i,a/c)\{lm|uv|ik\}.\end{aligned} \tag{A 9}$$

The coefficient c_{rr} is

$$c_{rr}(l,x) = \frac{2}{x^2}\rho\sigma((l^2-1)j_l(x) - x(l+1)j_{l+1}(x))l. \tag{A 10}$$

The coefficient $\Lambda_{lm,uv,ik}$ required in equation (75) is given by

$$\begin{aligned}\Lambda_{lm,uv,ik} = &-\frac{i(i-1)a^i(cQ_1(i,a/c) + aQ_2(i,a/c))}{\sigma\rho(-2(i^2-1)c^2 + (2i+1)acQ_1(i,a/c) - a^2Q_2(i,a/c))}\langle lm|uv|ik\rangle\\ &+/\frac{a^i(-3cQ_1(i,a/c) + aQ_2(i,a/c))}{\sigma\rho(-2(i^2-1)c^2 + (2i+1)acQ_1(i,a/c) - a^2Q_2(i,a/c))}\langle\langle lm|uv|ik\rangle\rangle\\ &+\left(-\frac{j_l(a/c)a^ic^2D_{rr}(i,a/c)}{a^3c_{\Theta r}(l,a/c)} + \frac{j_l(a/c)a^ic^2D_{\Theta\Theta 1}(i,a/c)}{a^3c_{\Theta r}(l,a/c)}\right)\langle\langle\langle lm|uv|ik\rangle\rangle\rangle\\ &-\frac{j_l(a/c)a^ic\left(\frac{\partial}{\partial x}D_{\Theta r}(i,x)\right)_{x=a/c}}{a^2c_{\Theta r}(l,a/c)}\langle\langle\langle lm|ik|uv\rangle\rangle\rangle\\ &-\frac{j(l,a/c)a^ic^2D_{\Theta\Theta 2}(i,a/c)}{a^3c_{\Theta r}(l,a/c)}\langle lm|ui|ik\rangle.\end{aligned} \tag{A 11}$$

Here we used the abbreviation $Q_i(l,x)$ for the ratio of the different spherical Bessel functions

$$Q_i(l,x) = \frac{j_{l+i}(x)}{j_l(x)}. \tag{A 12}$$

The coefficient $\langle\langle lm|uv|ik\rangle\rangle$ is defined by

$$\langle\langle lm|uv|ik\rangle\rangle = -\tfrac{1}{2}(l(l+1) - u(u+1) - i(i+1))\langle lm|uv|ik\rangle. \tag{A 13}$$

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